Method of Alternating Projections for Solving Absolute Value Equations

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Joint work with Jein-Shan Chen and Matthew K. Tam

Outline

Absolute value equation (AVE)

$$Ax = c \tag{LS}$$

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The system of equations

$$Ax + B|x| = c \qquad \text{(AVE)}$$

where $A, B \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^m$ is called an absolute value equation.

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where $A, B \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$ is called an absolute value equation.

- This reduces to a system of linear equations when $B = 0$.
- When $m = n$, (AVE) $\Longleftrightarrow$ (LCP)[2]

$$\boxed{x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad \langle x, Mx + q \rangle = 0} \qquad \text{(LCP)}$$

known as the linear complementarity problem.

[2] O. L. Mangasarian, R.R. Meyer, Absolute value equations, *Linear Algebra and its Applications*, 419, 359–367, 2006.

Known methods for solving AVEs

Case I. $m = n$ and $B = -I$
There are plenty of algorithms for

$$Ax - |x| = c$$

but they can be roughly classified as:

- Newton-based methods. Semismooth Newton, Inexact Newton, and smoothing Newton approaches.
- Picard iterations. When A is invertible, solutions of (AVE) corresponds to fixed points of $T(x) := A^{-1}(|x| + c)$.
- Matrix splitting method. Splitting strategies for A to reduce cost of each iteration instead of solving a full linear system.
- Successive linearization algorithm. Reformulate (AVE) as a concave minimization problem, solved by successive linearization.

Case II (General case). $m \neq n$, $B \neq I$

- Only successive linearization algorithm is known to handle the general case.
- **Note:** The interest to this case might be purely theoretical only, as there are no known applications (yet).

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$$\begin{aligned} S_1 &:= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : Ax + By = c\} \quad &\text{(affine)} \\ S_2 &:= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : y = |x|\} \quad &\text{(nonconvex)} \end{aligned}$$

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- We obtain a nonconvex feasibility problem:

$$\boxed{\text{Find} \quad (x, y) \in S_1 \cap S_2}$$

where $S_1, S_2 \subset \mathbb{R}^n$ are given by

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How do we solve a two-set feasibility problem?

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Solution methods for feasibility problems

- Classical approaches use projections: Given a nonempty set S, the projector onto S is given by

$$P_S(z) := \{s \in S : \|s - z\| \leq \|t - z\| \quad \forall t \in S\}.$$

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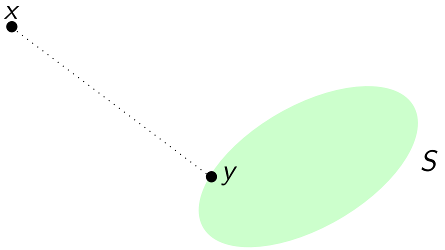
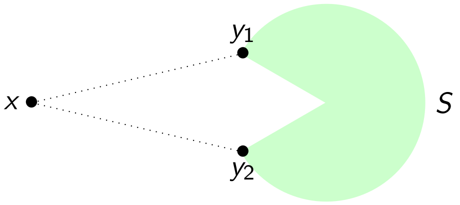
- When S is convex and closed, P_S is single-valued everywhere.
- When S is nonconvex, P_S could be multivalued.

Projectors onto convex and nonconvex set



$$P_S(x) = \{y\}$$



$$P_S(x) = \{y_1, y_2\}$$

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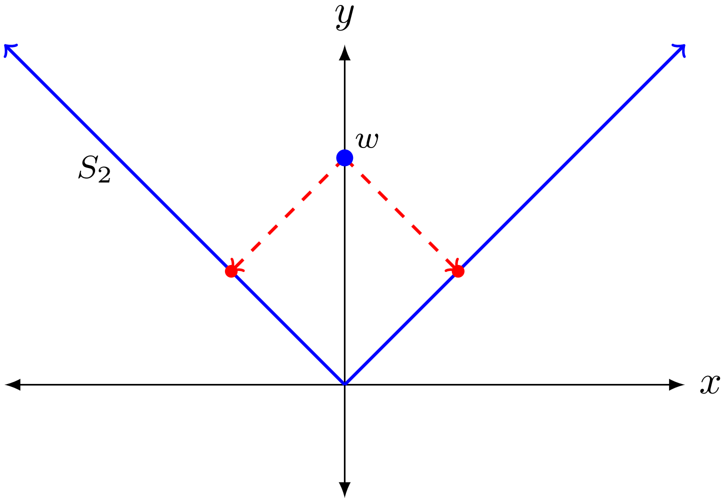

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1. Method of alternating projections (MAP)

$$z^{k+1} \in P_{S_1}(P_{S_2}(z^k)), \qquad k = 0, 1, 2, \ldots$$

2. Method of averaged projections (MAveP)

$$z^{k+1} \in \frac{P_{S_1}(z^k) + P_{S_2}(z^k)}{2}, \qquad k = 1, 2, \ldots$$

3. Douglas-Rachford method (DR)

$$z^{k+1} \in \frac{z^k + R_{S_1}(R_{S_2}(z^k))}{2}, \quad k = 0, 1, 2, \ldots$$

where $R_S := 2P_S - Id$.

MAP, MAveP, DR

- Global convergence to $S_1 \cap S_2$ is known when the sets S_1 and S_2 are both closed and convex.
- Nonconvex case is problematic.

Our approach

We apply **MAP** to

$$\boxed{\text{Find} \quad (x, y) \in S_1 \cap S_2}$$

with

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2. Is MAP globally/locally convergent?
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1. If a generated MAP sequence is convergent, is the limit a solution? (Focus of this work)
2. Is MAP globally/locally convergent?
3. Do we obtain good numerical results?

Outline

Is the limit (if it exists) always a solution?

$S_1 = \{(x, y) \in \mathbb{R}^2 : x - y = -2/\sqrt{2}\}$
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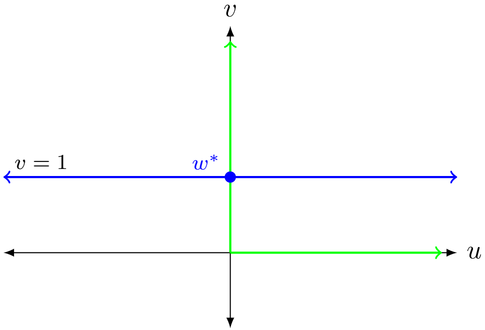

Figure: C_1 := blue line
C_2 := nonnegative (u, v)-axes

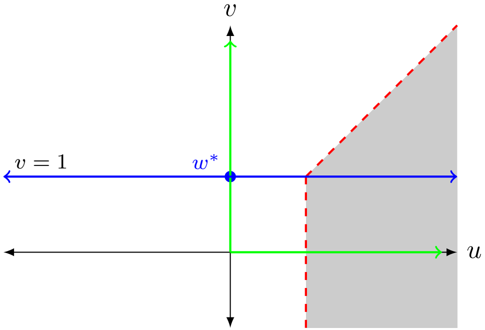

Figure: $C_1 :=$ blue line; $C_2 :=$ nonnegative (u, v)-axes

Location of initial point	**Limit of** $(P_{C_1} \circ P_{C_2})^k$
Gray region	Not a solution
Red dashed line	Depends on selected element of P_{C_2}
Else	Solution

Fixed points

Definition

The set of fixed points of MAP are given by

$$\mathrm{Fix}(P_{S_1} \circ P_{S_2}) = \{z \in \mathbb{R}^n \times \mathbb{R}^n : z \in (P_{S_1} \circ P_{S_2})(z)\}, \tag{1}$$

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- For the feasibility reformulation of the AVE:

Which fixed points belong to $S_1 \cap S_2$?

Change of Variables

- Let R be the orthogonal matrix $R = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & -I_n \\ I_n & I_n \end{bmatrix}$ and let $w = R^\top z$, where

$$\begin{aligned} z &= (x, y) && \text{(original variables)} \\ w &= (u, v) && \text{(new variables)} \end{aligned}$$

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- The constraint sets S_1 and S_2 become

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- $\boxed{\text{Find } z \in S_1 \cap S_2} \iff \boxed{\text{Find } w \in C_1 \cap C_2}$
- $R^{\mathsf{T}}\,\mathrm{Fix}(P_{S_1} \circ P_{S_2}) = \mathrm{Fix}(P_{C_1} \circ P_{C_2})$

Case I: Arbitrary m and n

Denote

$$\hat{C}_2 = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : u_i v_i = 0 \ \forall i \in [n]\},$$
$$\Omega = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : \text{for each } i \in [n], u_i \geq 0 \text{ or } v_i \geq 0\}.$$

Theorem (A, Chen & Tam, 2022, JFPTA)

Let $T = [\, A + B \quad -A + B] \in \mathbb{R}^{m \times 2n}$. If

$$\text{Ker}(T)^\perp \cap \hat{C}_2 = \{0\}, \tag{C}$$

then for any $c \in \mathbb{R}^m$,

$$\text{Fix}(P_{C_1} \circ P_{C_2}) \cap \Omega = C_1 \cap C_2.$$

Questions

1. When does condition (C):

$$\operatorname{Ker}(T)^{\perp} \cap \hat{C}_2 = \{0\}, \tag{C}$$

   hold?

2. Under what assumptions do we get

$$\operatorname{Fix}(P_{C_1} \circ P_{C_2}) \subset \Omega?$$

Case II: $m = n$

A matrix Q is said to be nondegenerate if all its principal minors are nonzero[3].

[3] That is, $\det(Q_{\Lambda\Lambda}) \neq 0$ for all $\Lambda \subset \{1, \ldots, n\}$

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Theorem (A, Chen & Tam, 2022, JFPTA)

If $Q := (A^\top + B^\top)(A^\top - B^\top)^{-1}$ is nondegenerate[4], then

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Nondegeneracy of Q holds, for instance, when $\sigma_{\min}(A) > \sigma_{\max}(B)$ or $\sigma_{\max}(A) < \sigma_{\min}(B)$.

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Example 1: Importance of nondegeneracy

- Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = -I$ and $c = (-10, -19)/\sqrt{2}$.
- Then
$$Q = \begin{pmatrix} -1.5 & 1.5 \\ 1 & 0 \end{pmatrix}$$
is degenerate.
- Let $\bar{w} = (-0.9231, 4.7026, 9.0872; 0.6154)$. Then
$$\bar{w} \in \mathrm{Fix}(P_{C_1} \circ P_{C_2}) \cap \Omega \quad \text{and} \quad \bar{w} \notin C_1 \cap C_2.$$

Case II: $m = n$ (continued)

A matrix Q is said to be a P-matrix if all of its principal minors are positive[5].

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If $\sigma_{\min}(A) > \sigma_{\max}(B)$, then Q is positive definite. Thus, (2) holds.

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Outline

Local convergence

Theorem (A, Chen & Tam, 2022, JFPTA)

Suppose $w^* \in C_1 \cap C_2$. Then there exists sufficiently small $\delta > 0$ such that for any w^0 with $\|w^0 - w^*\| < \delta$, any generated MAP sequence converges to **a point** in $C_1 \cap C_2$.

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- **Proof 2:** Using an optimization reformulation of the feasibility problem.
- **By-product of Proof 2:** Global convergence for *homogeneous AVE*.

Linear rates: Arbitrary m and n

Proposition (A, Chen & Tam, 2022, JFPTA)

If condition (C) holds:

$$\text{Ker}(T)^{\perp} \cap \hat{C}_2 = \{0\}, \tag{C}$$

and $w^* \in C_1 \cap C_2$ such that $(u_i^*, v_i^*) \neq (0,0)$ $(\forall i)$, then any sequence generated by MAP with initial point sufficiently close to w^* converges linearly to **a point** in $C_1 \cap C_2$.

This is a consequence of Lewis, Luke and Malick's linear convergence results for super-regular sets with linearly regular intersection.

Linear rates: $m = n$

Theorem (A, Chen & Tam, 2022, JFPTA)

If Q is nondegenerate and $w^* \in C_1 \cap C_2$ such that $(u_i^*, v_i^*) \neq (0,0)$ $(\forall i)$, then any sequence generated by MAP with initial point sufficiently close to w^* converges linearly to w^*.

Global convergence

- We have global convergence for
  1. homogeneous AVE
  2. Relaxed version of MAP:

$$w^{k+1} \in (1-\gamma)P_{C_2}(w^k) + \gamma(P_{C_1} \circ P_{C_2})(w^k), \quad \gamma \in (0,1)$$

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- No global convergence result for full MAP[6] .

[6] Not until our most recent work:
J.H. Alcantara and C.-p. Lee, Global convergence and acceleration of fixed point iterations of union upper semicontinuous operators: proximal algorithms, alternating and averaged nonconvex projections, and linear complementarity problems, arXiv:2202.10052, 2022.

Global convergence

- We have global convergence for
  1. homogeneous AVE
  2. Relaxed version of MAP:

$$w^{k+1} \in (1-\gamma)P_{C_2}(w^k) + \gamma(P_{C_1} \circ P_{C_2})(w^k), \quad \gamma \in (0,1)$$

- No global convergence result for full MAP[6] .
- **Conjecture:** Nondegeneracy is necessary for global convergence.

Outline

Example 1: $m = n$

- Set $A = A'/(t\sigma_{\min}(A'))$ with $a'_{ij} \sim U(-10, 10)$ and $t \sim U(0, 1)$.
- Set $x^* \in \mathbb{R}^n$ such that $x^*_i = r \cdot 10^{\alpha s}$ with $\alpha \in \{0, 1, 2, 3\}$, $r \sim U(-1, 1)$ and $s \sim U(0, 1)$.
- $c = Ax^* + B|x^*|$ with $B = -I$.
- $n = 5000$

Table: Results for Example 1

Method		α			
		0	1	2	3
MAP	Success rate	**1**	**0.99**	**0.87**	**0.62**
	Ave. Time	2.58	3.03	3.13	10.42
	Ave. Iter	40.85	52.51	55.44	250.39
GNM	Success rate	0.76	0.55	0	0
	Ave. Time	2.23	2.29	–	–
	Ave. Iter	3.93	4.00	–	–
PIM	Success rate	0.75	0.54	0.01	0
	Ave. Time	0.57	0.59	0.84	–
	Ave. Iter	4.99	5.65	22.00	–

GNM: Generalized Newton Method (Mangasarian, 2008)

PIM: Picard Iteration Method (Rohn, Hooshyarbaksh, and Farhadsefat, 2014)

Example 2: $m \neq n$

- Sample entries of $A, B \in \mathbb{R}^{m \times n}$ and $x^* \in \mathbb{R}^n$ from the standard normal distribution.
- Set $c = Ax^* + B|x^*|$
- $n = 500$
- $m = rn$ with $r \in \{0.25, 0.5, 0.75, 1.5, 2.0, 3.0\}$.

Table: Results for Example 2

Method		r					
		0.25	0.5	0.75	1.5	2	3
MAP	Ave. Time	0.01	0.03	0.26	0.12	0.02	0.19
	Ave. Iter	104.19	296.34	2162.84	227.16	1	1
SLA	Ave. Time	4.21	19.69	63.60	26.11	31.33	90.31
	Ave. Iter	2.38	3.64	6.11	1	1	1

SLA: Successive linearization algorithm (Mangasarian, 2007)

Thank you for listening!

Some references

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